Analysis I Lecture 20

This wednesday:

1st Hour: Open question example

2nd Hour: Mock exam review

Last time:

Higher derivatives and C^k-functions:

We define $f^{(n)}$ inductively:

If $f^{(n-1)}$ defined then $f^{(n)} = \left(f^{(n-1)}\right)'$.

The set C^k of functions:

$$C^k = \left\{ f \;\middle|\; f', f'', \dots f^{(k)} \text{ exist and continuous} \right\} =$$

$$= \left\{ f \in C^{k-1} \;\middle|\; f^{(k-1)} \text{ is continuously differentiable} \right\}$$

i.e. $\left(f^{(k-1)}\right)'$ exists and continuous

$$C^\infty = \bigcap_{k=0}^{\infty} C^k = \left\{ f \;\middle|\; f^{(n)} \text{ exists for every } n \in \mathbb{N} \right\}$$

We use $C^k(I)$ or $C^\infty(I)$ for C^k or C^∞ functions on I.

Example $f(x) = e^x$ $f: \mathbb{R} \to \mathbb{R}$

$f'(e^x) = e^x$ so e^x is continuously differentible so it is C^1.

In fact $f^{(n)}(x) = e^x$ for any $n \in \mathbb{N}$ so $e^x \in C^n(\mathbb{R})$

$\Rightarrow$ e^x is $C^\infty(\mathbb{R})$

Example $f: \mathbb{R} \to \mathbb{R}$ $\quad f(x) = |x| \cdot x$

One can show that:

$$f'(x) = 2|x|$$

To show notice that:

$$f(x) = \begin{cases} x^2 & \text{for } x > 0 \\ 0 & \text{for } x = 0 \\ -x^2 & \text{for } x < 0 \end{cases} \Rightarrow f'(x) = \begin{cases} 2x & x > 0 \\ 0 & x = 0 \\ -2x & x < 0 \end{cases}$$

So we get that

$f'(x)$ is continuous

$\Rightarrow f \in C^1(\mathbb{R})$.

But $f''(x)$ doesn't exist

since $2|x|$ is not differentiable

at 0 $\Rightarrow f \notin C^2(\mathbb{R})$

Remark $|x|\cdot x$ is C^∞ if we consider it on $\mathbb{R}\setminus\{0\}$

$$|x|\cdot x \in C^\infty(\mathbb{R}\setminus\{0\}).$$

Example

Caution!

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Here f is differentiable but f' is not continuous

$\Rightarrow f \notin C^1$

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

for $x \neq 0$ we can compute $f'(x)$ by standard technique:

$$f'(x) = 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right), \quad x \neq 0$$

To compute $f'(0) =$

$$= \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{h^2 \cdot \sin\left(\frac{1}{h}\right)}{h} =$$

$$= \lim_{h \to 0} h \cdot \underbrace{\sin\left(\frac{1}{h}\right)} = 0.$$

In total:

$$f'(x) = \begin{cases} 2x \cdot \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Notice that $\lim\limits_{x \to 0} f'(x)$ does not exists since $\Rightarrow f'$ is not cont. $\blacksquare$

Today:

Local and Global Extreme

Min or Max

Rolle's theorem

Mean value theorem

Monotonicity and derivative

from full course notes.

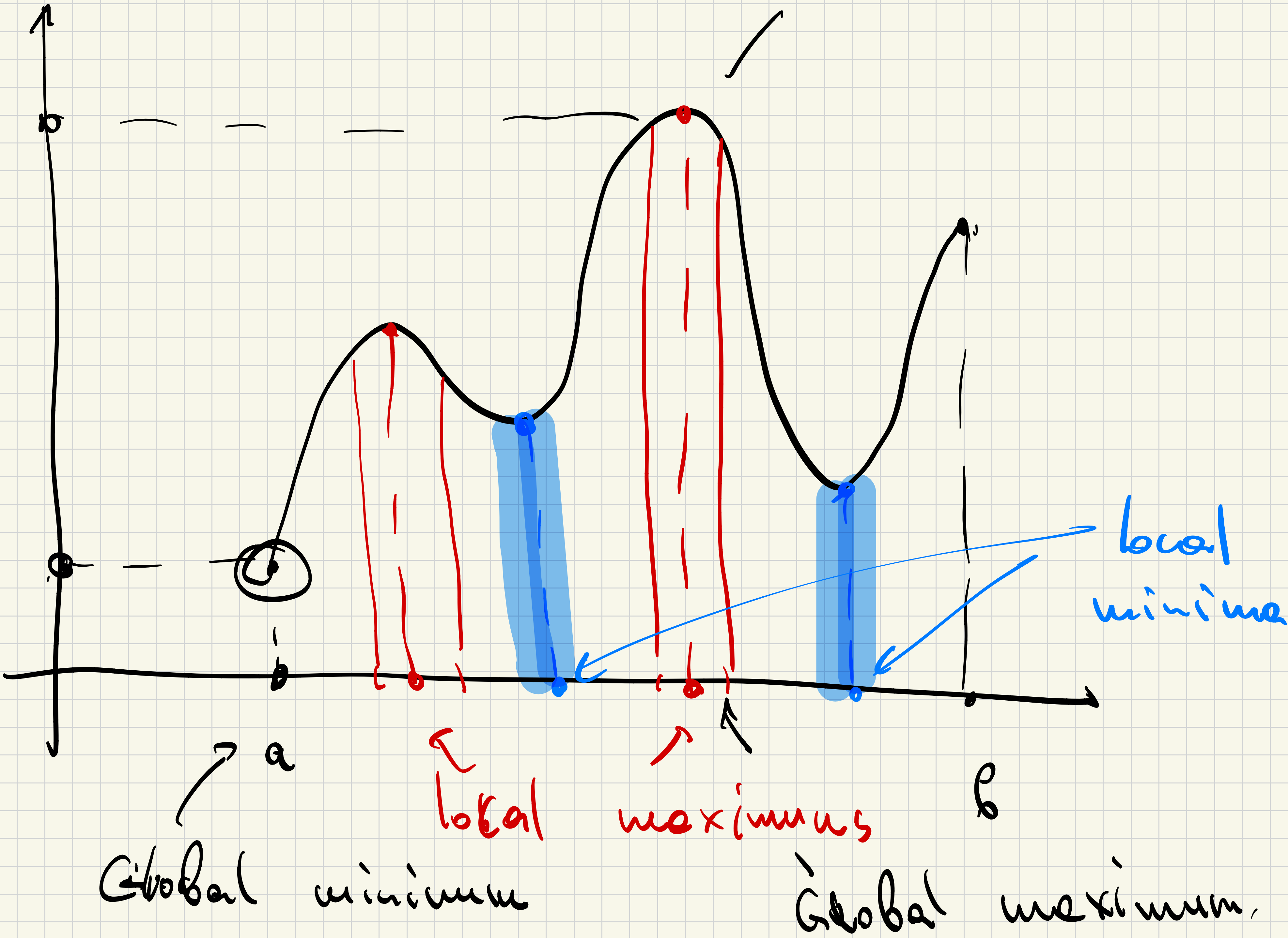
Local / Global Extreme

Definition 7.48 Let $f: E \to \mathbb{R}$ be a function. Then $x_0 \in E$ is a local maximum (minimum) if $\exists\ \delta > 0$ s.t. $(x_0 - \delta, x_0 + \delta) \subset E$ and $f(x_0) \geq f(x)$ $\left(f(x_0) \leq f(x) \right)$

$$\forall\ x \in (x_0 - \delta, x_0 + \delta).$$

2. x_0 is called global maximum (minimum) if $f(x_0) \geq f(x)$ ($f(x_0) \leq f(x)$) $\forall \; x \in E$.

3. x_0 is called local (global) Extremum if it is a local (global) minimum and maximum


a
b
local maximums
local minima
Global minimum
Global maximum

Proposition 7.52

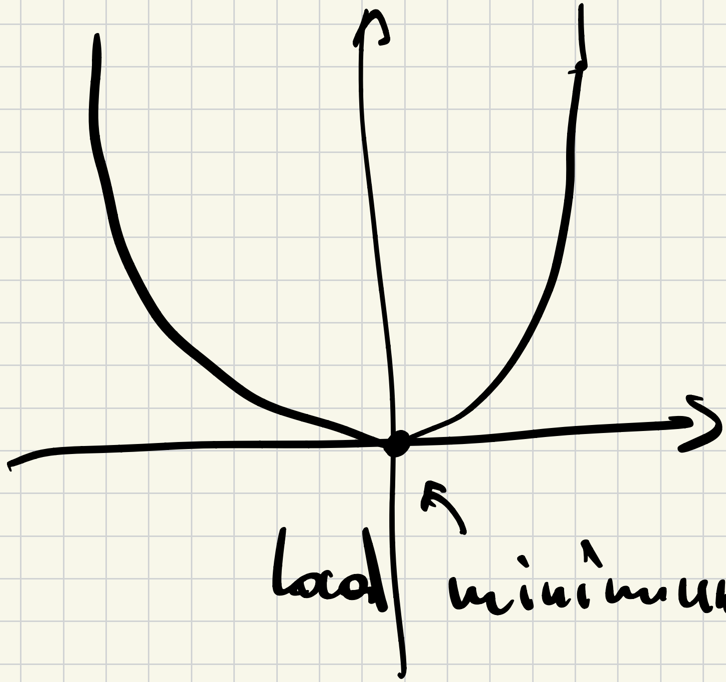
$f: E \to \mathbb{R}$ is differentiable at x_0 and x_0 is a local $\underbrace{\text{extremum}}$ then $f'(x_0) = 0$.

i.e. local minimum

or local maximum

Examples

- $f(x) = x^2$

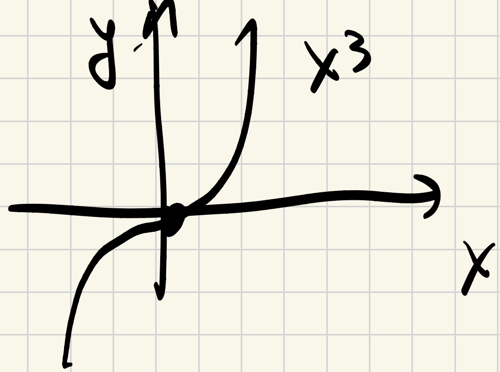


$f'(x) = 2x$

$f'(x) = 0 \iff x = 0$

- $f'(x_0) = 0 \;\not\Rightarrow\; x_0$ is local extremum

$f(x) = x^3$ $\quad f'(x) = 3x^2$ has 0 at $x = 0$



but: 0 is not local min / max

Proof of Prop 7.52

Assume that x_0 is a local maximum.

f is differentiable at $x_0 \Rightarrow$

$$f'_{left}(x_0) = f'(x_0) = f'_{right}(x_0).$$

$$\lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0}$$

$f(x) - f(x_0) \leq 0$ since x_0 is local maximum

$x - x_0 < 0$ since left limit

$\Rightarrow f'_{left}(x_0) \geq 0$.

$$f'_{right}(x_0) = \lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0} \le 0$$

$x - x_0 > 0$

since right limit

$$\Rightarrow \quad f'(x_0) \le 0 .$$

$$\Rightarrow \quad 0 \geq f'_{righ}(x_0) = f'(x_0) = f'_{left}(x_0) \geq 0$$

$$\Rightarrow \quad f'(x_0) = 0 \qquad \blacksquare$$

Definition 7.54 We say $x_0 \in E$ is a stationary point of f if f is differentiable at x_0 and $f'(x_0) = 0$.

If f is differentiable then

local max or min $\Rightarrow$ stationary point.

Question How to see if a stationary point is a local extremum?

Proposition (sufficient condition for stationary $\Rightarrow$ local extremum)

Assume that $f \in C^k(I)$ and such that $f'(x_0) = f''(x_0) = \dots = f^{(k-1)}(x_0) = 0$

And k is **even**.

Then if $f^{(k)}(x_0) > 0 \Rightarrow x_0$ is local min

if $f^{(k)}(x_0) < 0 \Rightarrow x_0$ is local max.

<u>Example</u> Consider $f(x) = x^2$

$x_0 = 0$ is local minimum:

$f'(x) = 2x$ $\qquad f'(0) = 0$

$f''(x) = 2$ $\qquad \underline{\underline{f''(0) = 2 > 0}}$

first non-zero derivative and it is even order

$\Rightarrow$ 0 is a local min.

Example x^3

$f'(x) = 3x^2$ $\qquad$ $f'(x_0) = 0$

$f''(x) = 6 \cdot x$ $\qquad$ $f''(x_0) = 0$

$f'''(x) = 6$ $\qquad$ $f'''(x_0) = 6 > 0$

But this is an odd derivative.

Finding global maximum / minimum

Let $f : [a,b] \to \mathbb{R}$ cont. then it has global maximum and minimum.

How to find? Assume that f is differentiable on (a,b).

- Find all stationary points $x_1 \dots x_k$
- compare values at $x_1 \dots x_k$ and a, b

Example $f(x) = x^3 - 2x^2 + x + 1$ on $[0,2]$

find global maximum and minimum of $f(x)$.

1st: find stationary points:

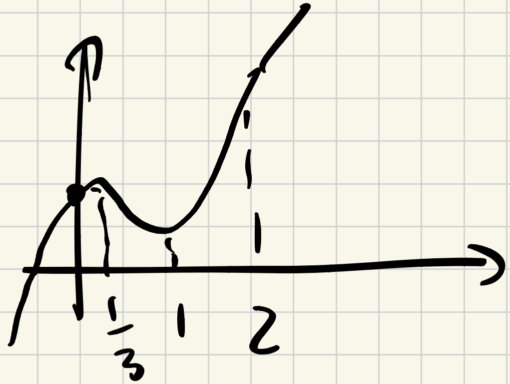
$$f'(x) = 3x^2 - 4x + 1$$

$$f'(x) = 0 \iff \begin{cases} x = \frac{1}{3} \\ x = 1 \end{cases}$$

Stationary points both in $[0,2]$.

2nd To find global min and max

Need to compare



end points

$f(0) = 1$

$f(2) = 3$

stationary points

$f\left(\frac{1}{3}\right) = \frac{31}{27}$

$f(1) = 1$

Global minima

Global maximum

Rolle's and Mean Value Theorem

Theorem 7.56 (Rolle's theorem)

Let $f : [a,b] \to \mathbb{R}$ be continuous and differentiable on (a,b).

Assume that $f(a) = f(b)$, then

$\exists \; c \in (a,b)$ s.t. $f'(c) = 0$.

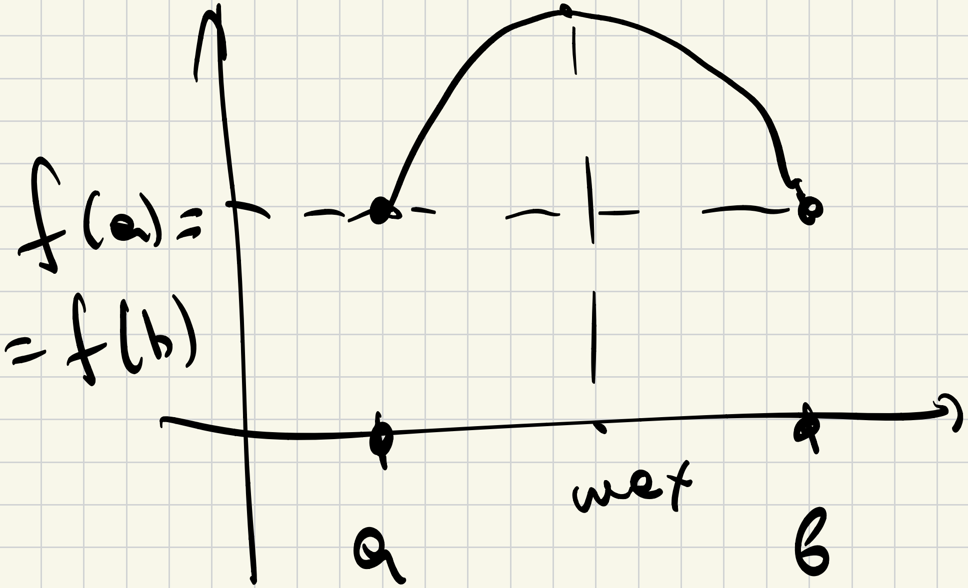
Proof. If f is a const. function,

$$f(x) = f(a) = f(b) \quad \forall x \in [a,b]$$

then $f'(x) = 0 \quad \forall x \in (a,b)$ and we are done.

- We can assume $f(x)$ is not constant.

Since f is cont. on $[a,b]$ it has a global max. and global minimum.



Since f is non constant at least one of the two will be in (a, b).

$\Rightarrow \exists\ c \in (a,b)$ which is a global (and hence local) extremum $\Rightarrow f'(c) = 0$.

$\blacksquare$

Mean value theorem

Theorem 7.58 (MVT) Let $f:[a,b]\to\mathbb{R}$ be continuous and differentiable on (a,b) then $\exists\ c\in(a,b)$ s.t.

$$\boxed{f'(c)=\frac{f(b)-f(a)}{b-a}}$$

Remark MVT $\Rightarrow$ Rolle's thm:

if $f(a)=f(b)\Rightarrow \dfrac{f(b)-f(a)}{b-a}=0$

<u>Pf.</u> In fact we can use Rolle's Theorem to show MVT:

Consider function:

$$g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b-a} \cdot (x-a)$$

$$\downarrow$$

$$g'(x) = f'(x) - \boxed{\frac{f(b) - f(a)}{b-a}}$$

$$g'(x) = 0 \iff f'(x) = \frac{f(b) - f(a)}{b - a}$$

but $g(a) = 0$

$g(b) = 0$ by direct calcu- lation

$\Rightarrow$ by Rolle's theorem $\exists\, c \in (a, b)$ s.t $g'(c) = 0$. ∎

Example

$$f : \left[0, \frac{\pi}{2}\right] \longrightarrow \mathbb{R}$$

$$f(x) = \sin^2 x \cdot e^x$$

Lemma 7.61 Corollary 7.60

Let $f, g, h: [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b).

1) Assume that $h'(x) = 0 \quad \forall x \in (a,b)$
then $h(x) = \text{const}$.

2) Assume that $f'(x) = g'(x) \quad \forall x \in (a,b)$
then $f(x) = g(x) + \text{const}$